Midterm this Wednesday

same rules as for midterm II

practice problems posted on website

Recall Taylor's Theorem

Assume $f:(a,b)\to\mathbb{R}$ s.t. its n-th derivative exists on (a,b)

Let $a<c<b$.

Then for each $x\in(a,b)$, $x\neq c$ we have

$$f(x)=\sum_{k=0}^{n-1}\frac{f^{(k)}(c)}{k!}(x-c)^k + R_n(x),$$

where $R_n(x)=\frac{f^{(n)}(y)}{n!}(x-c)^n$ for some y between x and c.

Question: Assume f infinitely many times differentiable in (a,b)

when is $f(x)$ given by its Taylor series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k \quad ?$$

Observation: It is given by its Taylor series $\Longleftrightarrow$ $\lim_{n \to \infty} R_n(x) \to 0$

Example 1 There are functions ∞-times differentiable for which Taylor series converges to $f(x)$ only at $x=c$!

- homework problem: $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$

- book: $f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$ } see book for details

Consider $f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$

<u>claim 1</u> f ∞-times differentiable for all $x \in \mathbb{R}$

$x \neq 0$: want to show: $\exists$ polynomial P_n of degree $\leq 3n$ such that $f^{(n)}(x) = e^{-1/x^2} P_n(1/x)$

proof by induction:

$n=1$: $f'(x) = e^{-1/x^2} \cdot -(-2)\, x^{-3}$

$= e^{-1/x^2} \cdot 2 \cdot (1/x)^3$ ✓

$n \to n+1$ $f^{(n+1)}(x) = \frac{d}{dx}\left(e^{-1/x^2} P_n(1/x) \right)$

$= e^{-1/x^2} \cdot 2 \cdot (1/x)^3 P_n(1/x)$

$+ e^{-1/x^2} \cdot P_n'(1/x) \cdot -\frac{1}{x^2}$ $=$

$$= e^{-1/x^2}\left[\underbrace{\frac{2}{x^3}P_n\left(\frac{1}{x}\right) - \frac{1}{x^2}P_n'\left(\frac{1}{x}\right)}_{=P_{n+1}\left(\frac{1}{x}\right)}\right]$$

[aside: can define polynomials P_n inductively by

$$P_0 = 1, \quad P_1(y) = 2y^3$$

$$P_{n+1}(y) = 2y^3 P_n(y) - y^2 P_n'(y) \qquad \left(y = \frac{1}{x}\right)$$

]

<u>$x=0$</u> need to use definition of differentiability!

$$f^{(n+1)}(0) = \left[f^{(n)}\right]'(0) \qquad \text{by ind. ass.}$$

$$= \lim_{x\to 0} \frac{f^{(n)}(x) - \overbrace{f^{(n)}(0)}^{=0}}{x-0}$$

$$= \lim_{x\to 0} \frac{f^{(n)}(x)}{x}$$

$$= \lim_{x\to 0} e^{-1/x^2} \underbrace{P_n(1/x) \cdot \frac{1}{x}}_{Q_n\left(\frac{1}{x}\right)}$$

Enough to prove the following <u>lemma</u>:

Let Q be a polynomial

$$\Rightarrow \lim_{x\to 0} e^{-1/x^2} Q\left(\frac{1}{x}\right) = 0$$

Proof. Apply l'Hospital's rule for

$$\lim_{x\to 0} \frac{Q(1/x)}{e^{1/x^2}} =$$

Observation: If $y = \frac{1}{x}$

$x \to 0 \Leftrightarrow y \to \infty$

$$= \lim_{y\to\infty} \frac{Q(y)}{e^{y^2}} = \lim_{y\to\infty} \frac{Q'(y)}{e^{y^2}\cdot 2y} = \lim_{y\to\infty} \frac{Q''(y)}{e^{y^2}(4y^2+2)} =$$

(aside: prove by ind. on n:

n-th derivative of e^{-y^2} is

$e^{-y^2} P_n(y)$ for some polynomial P_n)

proof. $(n+1)$-st derivative $= \frac{d}{dy}\left(e^{y^2} P_n(y)\right) =$

$$= e^{y^2} 2y P_n(y) + e^{y^2} P_n'(y)$$

$$= e^{y^2}\big(\underbrace{2y P_n(y) + P_n'(y)}_{P_{n+1}(y)}\big)$$

$\Rightarrow$ If Q is a polynomial of degree d

$$\Rightarrow \lim_{y \to \infty} \frac{Q(y)}{e^{y^2}} = \dots \lim_{y \to \infty} \frac{Q^{(d+1)}(y)}{e^{y^2}(P_{n+1}(y))} = 0$$

apply l'Hospital's rule $d+1$ time

$Q^{(d+1)}(y) = 0$

have shown:

$$f^{(n+1)}(0) = \lim_{x\to 0} \frac{f^{(n)}(x)}{x}$$

$$= \lim_{y\to\infty} e^{-y^2} Q_n(y)$$

$$= 0$$

Conclusion: $f^{(n)}(0) = 0$ for all n

$\Rightarrow$ Taylor series of f at $c=0$ is equal to

$$\sum_{k=0}^{\infty} 0 \cdot x^k = 0 \quad \text{for all } x!$$

on the other hand:

$$e^{-1/x^2} \neq 0 \quad \text{for } x \neq 0$$

$\neq$ value of its Taylor series for $x \neq 0$

Problem from practice exam:

Let $f(x) = (1+x)^{3/2}$

Show: $1 + 3/2\, x < f(x) < 1 + \frac{3}{2}x + \frac{3}{4}x^2$ for $x > 0$

prove yourself

Use Taylor's Theorem!

$f'(x) = \frac{3}{2}(1+x)^{1/2}$

$f''(x) = \frac{3}{2}\frac{1}{2}(1+x)^{-1/2}$

$f'''(x) = \frac{3}{2}\cdot\frac{1}{2}\cdot\frac{-1}{2}(1+x)^{-3/2}$

$f'(0) = \frac{3}{2}$

$f''(0) = \frac{3}{4}$

n=2:

$$f(x) = f(0) + f'(0)x + R_2(x)$$
$$= 1 + \frac{3}{2}x + R_2(x)$$

in order to prove $<$ enough to show: $\boxed{R_2(x) > 0}$!

By Taylor's Theorem, there exists $y \in (0, x)$ such that

$$R_2(x) = \frac{f''(y)}{2!} x^2$$

$$f''(y) = \frac{3}{4}(1+y)^{-1/2} > 0 \quad \text{for } y \in (0, x)$$

$$\Rightarrow \quad R_2(x) > 0$$

$$\Rightarrow \quad f(x) = 1 + \frac{3}{2}x + R_2(x) > 1 + \frac{3}{2}x .$$